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B.Sc. Sem-II, MSC-02

Three mutually perpendicular generators.

To find the necessary and sufficient condition that a cone may have three mutually perpendicular generators.

Also, To prove that if a cone has one set of three mutually perpendicular generators, it has an infinite number of such sets.

Let the equation of the cone be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{—— (1)}$$

Firstly we shall derive the necessary condition

Let us suppose that the cone possesses three mutually perpendicular generators.

Let us transform the equation of the cone taking three mutually perpendicular generators as the co-ordinate axes. Then the equation of the cone in the new co-ordinate system will be of the form.

$$f'yz+g'zx+h'xy=0$$

as the cone now passes through the new co-ordinate axes

or $$0.x^2+0.y^2+0.z^2+2f'yz+2g'zx+2h'xy=0 \quad \text{—— (2)}$$

From (1) and (2) by the principle of invariants

$$a+b+c=a'+b'+c'=0+0+0=0$$

Hence $$a+b+c=0$$

Thus the necessary condition is achieved

Now, we shall prove that the condition is sufficient

Let $$a+b+c=0 \quad \text{—— (3)}$$

Let us take any generator of the cone as the new x-axis

Then the transformed equation of the cone will be of the form

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0 \quad \text{—— (4)}$$

Since the new x-axis is a generator of the cone (4)
therefore the direction cosines 1, 0, 0 of the new x-axis must
satisfy (4)

$$\therefore \quad a'=0 \quad \text{—— (5)}$$

By the principle of invariants

$$a'+b'+c'=a+b+c=0 \quad \text{by (3)}$$

$$\therefore \quad \text{by (5)} \quad b'+c'=0 \quad \text{—— (6)}$$

Now the plane through the vertex $(0,0,0)$ of the cone
perpendicular to the new x-axis is the new YZ plane that
is $x=0$

It meets the cone (4) in lines given by

$$x=0, \; b'y^2+c'z^2+2f'yz=0 \quad \text{—— (7)}$$

But by (6) $b'+c'=0$ therefore the equation (7) gives a pair
of perpendicular lines in the YZ-plane.

Hence the cone has three mutually perpendicular generators.

Since the new x-axis is choosen arbitrarily therefore
the cone must have an infinite number of sets of
three mutually perpendicular generators.

Note :– (Principle of invariants) If by transformation from
one set of rectangular axes to another with the
same origin

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

transforms into

$$a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy$$

then,

$$a+b+c=a_1+b_1+c_1$$